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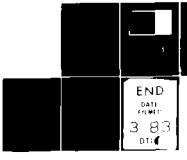

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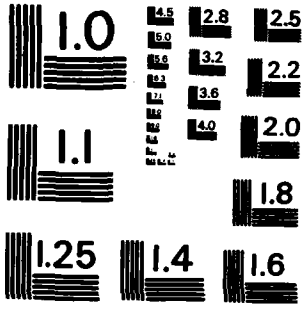

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ON THE DOMAIN SPACE FOR CONSTITUTIVE LAWS IN LINEAR VISCOELASTICITY

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ABSTRACT

We characterize those constitutive laws in linear viscoelasticity which are compatible with certain phenomenological conditions.

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Work Unit Number 2 - Physical Mathematics

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SIGNIFICANCE AND EXPLANATION

The constitutive law of a linearly viscoelastic fluid has the form

$$\underline{T} = \int_{-\infty}^{t} a(t-s)\underline{\gamma}(t,s)ds$$

where $\underline{T}$ is the stress, $\underline{\gamma}$ is the linearized relative strain, and the kernel a is the memory function. This kernel a may in general be a distribution. It is assumed that the following phenomenological conditions hold:

1. The fluid resists deformation; positive strains yield positive stresses.
2. The strain resulting from a more recent instant of time has a greater influence than that from a more remote time.

It is shown that the only kernels a consistent with these conditions consist of α) a positive, monotone decreasing function defined on $(0,\infty)$ and β) a δ'-distribution located at 0. The latter form of the kernel a corresponds to a Newtonian fluid.

ON THE DOMAIN SPACE FOR CONSTITUTIVE LAWS IN LINEAR VISCOELASTICITY

Michael Renardy

1. Introduction

The most general constitutive law for an isotropic, incompressible linearly viscoelastic fluid has the form

$$\underline{\underline{T}}(t) = \int_{-\infty}^{t} a(t-s)\underline{\underline{\gamma}}(t,s)ds, \tag{1.1}$$

where $\underline{\underline{T}}$ denotes the extra stress tensor, and $\underline{\underline{\gamma}}$ denotes the linearized relative strain tensor.

A recent paper by Saut and Joseph [4] addresses the question of specifying suitable domain spaces for the history of $\underline{\underline{\gamma}}$. The class of "admissible" kernels a is then given by the dual of the domain space. Depending on the choice of domain space, the kernels may be functions or, more generally, distributions. Saut and Joseph [4] discuss various choices for the domain and characterize the resulting dual spaces.

In this paper, a different point of view is adopted. Rather than specifying a domain space at the outset, I shall investigate the implication of certain phenomenological restrictions, which are commonly accepted in rheology. It will turn out that these restrictions alone guarantee a certain regularity of the kernel a, thus implying that the constitutive law is defined on a certain minimal domain space.

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We adopt the following two conditions, which are easily motivated by physical intuition (to my knowledge, they were first formulated by Boltzmann [1]):

1. The fluid resists deformation; positive strains lead to positive stresses.
2. The strain resulting from a more recent instant of time always has a greater influence than that resulting from a more distant time.

If a were a smooth function, these conditions would be written as $a > 0$, $a' < 0$. We adopt a generalized form of this, which is made precise below.

We shall find that the only admissible non-function contribution to the kernel is that which corresponds to the Newtonian fluid. In particular, fluids of higher grade are excluded (they do not satisfy condition 1).

In chapter 3 I investigate the stability of fluids of higher grade. Expanding an argument of Joseph [3], I find that no fluid of grade higher than four can have a stable rest state. Moreover, if the domain occupied by the fluid is infinite, the initial value problem for such fluids is ill-posed. Therefore, even for smooth initial histories, such fluids cannot be used to approximate unsteady flows.

2. Characterization of Admissible Kernels

A priori it will only be assumed that the constitutive law is defined on the following space of test functions:

$$E = \{\varphi \in C^{\infty}[0, \infty),\ \text{supp}\ \varphi\ \text{is compact},\ \varphi(0) = 0\}.$$

Thus the mapping $\varphi \to \int_0^{\infty} a(s)\varphi(s)ds$ is a continuous linear functional on E. The conditions 1 and 2 above are reformulated in the following distributional form:

1^*. $\int_0^{\infty} a(s)\varphi(s)ds \geq 0$ for $\varphi \in E$ such that $\varphi \geq 0$.

2^*. $\int_0^{\infty} a(s)\varphi'(s)ds \geq 0$ for $\varphi \in E$ such that $\varphi \geq 0$. supp $\varphi \subset (0,\infty)$ compact.

These conditions alone guarantee that the kernel a can actually be defined on a much larger domain space than E.

Theorem:

If 1^*, 2^* hold, then a can be extended to a linear functional defined on the space

$$F = \{\varphi \in L^1(0,\infty),\ \varphi(0) = 0 \ \text{and}\ \varphi \ \text{is differentiable at 0}\}.$$

This extension is continuous in the sense that for any $\alpha > 0$ there is a constant $C(\alpha)$ such that

$$\left|\int_0^{\infty} a(s)\varphi(s)ds\right| \leq C(\alpha)\left\{\int_0^{\infty} |\varphi(s)|ds + \sup_{s\in[0,\alpha]} \frac{|\varphi(s)|}{s} + |\varphi'(0)|\right\}$$

(If $\varphi \in F$, then the right hand side of this inequality is finite for α small enough).

Proof:

Let us first consider how the kernel a acts on test functions φ supported on an interval $[\alpha,\beta]$, $\alpha > 0$. For any such interval there is a function $\psi_{\alpha\beta} \in E$ such that $\psi_{\alpha\beta} > 0$ and $\psi_{\alpha\beta} = 1$ on the interval $[\alpha,\beta]$. We thus have

$$-\psi_{\alpha\beta}(s) \cdot \max_{t\in[\alpha,\beta]} |\varphi(t)| \leq \varphi(s) \leq \psi_{\alpha\beta}(s) \cdot \max_{t\in[\alpha,\beta]} |\varphi(t)|.$$

From condition 1^*, it follows that

$$\left|\int_0^\infty a(s)\varphi(s)ds\right| \leq \max_{t\in[\alpha,\beta]} |\varphi(t)| \cdot \int_0^\infty a(s)\psi_{\alpha\beta}(s)ds,$$

whenever supp φ is contained in $[\alpha,\beta]$. Hence, the kernel a can be extended to a continuous linear functional on the space $C^0[\alpha,\beta]$. According to a well known result [2], this implies that $a(s)ds = d\mu(s)$, where μ is a finite regular countably additive measure on $[\alpha,\beta]$. Because of condition 2^*, a' (the generalized derivative of a) has the same property. This implies that a is a function of bounded variation. It is easy to conclude from 1^* that a is positive. Moreover, we have $-a'(s)ds = d\bar{\mu}(s)$ for a certain regular countably additive measure $\bar{\mu}$. The characteristic function χ_I of any open interval $I \subset (\alpha,\beta)$ can be approximated by a sequence of positive test functions. Property 2^* and Lebesgue's theorem guarantee that $\bar{\mu}(I)$ is positive. Hence $\bar{\mu}$ is a positive measure, which implies that a is monotone decreasing. (We have $a(t) - a(t') = -\bar{\mu}(t,t')$ for almost every t,t').

Let us now consider test functions φ supported on $[0,\alpha]$. Let $\psi_\alpha \in E$ be such that $\psi_\alpha(s) = s$ for $s \in [0,\alpha]$ and $\psi_\alpha \geq 0$. Then

$$-\psi_\alpha \max_{t\in[0,\alpha]} |\varphi'(t)| \leq \varphi \leq \psi_\alpha \max_{t\in[0,\alpha]} |\varphi'(t)|.$$

Therefore, we have $\int_0^\alpha a(s)\varphi(s)ds = \int_0^\alpha b(s)\varphi'(s)ds$, and $b(s)ds = d\nu(s)$ for a finite, regular, countably additive measure ν. Since we already know how the kernel a acts on functions supported away from 0, we need only be concerned with the possibility that ν has an atom at 0. We thus obtain

$$\int_0^\infty a(s)\varphi(s)ds = C\,\varphi'(0) + \int_0^\infty \hat{a}(s)\varphi(s)ds \quad \text{for} \quad \varphi \in E. \tag{2.1}$$

where $\hat{a}$ is a positive, monotone decreasing function defined on $(0,\infty)$. Condition 1^* implies that $C \geq 0$, and for the integral to be defined on E it is necessary and sufficient that $\hat{a}(s)\cdot s$ is locally integrable. (Since $\varphi \in C^\infty$, $\varphi(0) = 0$, we have in general $\varphi(s) \sim s$ as $s \to 0$). From here the theorem is immediate.

Remark:

When speaking of "domain spaces", we have regarded (1.1) as a functional, i.e. as a mapping of histories of $\underline{y}$ to present values of $\underline{x}$. A different sometimes useful view of convolution integrals such as (1.1) is that of an operator, i.e. a mapping of histories of $\underline{y}$ to histories of $\underline{x}$. With this interpretation, the domain space for the relation (2.1) will be different from F (of course, it depends on the space that the history of $\underline{x}$ is required to be in).

Remark on the formulation of constitutive equations:

I have shown that, if Boltzmann's theory is reformulated in a distributional setting, then the Newtonian fluid appears as a special case. Constitutive models that combine memory integrals and Newtonian contributions (or nonlinear modifications thereof) are therefore natural. Such models are usually used for polymer solutions with Newtonian solvents and have also been suggested for polymer melts. On the other hand, fluids of higher grade, when taken as "exact" models rather than singular perturbation expansions, are incompatible with Boltzmann's postulates.

3. Nonexistence of "materials of the differential type" [5]

As we have seen, fluids of grade higher than 1 are not consistent with Boltzmann's conditions. In this chapter I show that, in addition, fluids of grade higher than 4 can not be stable at rest if the domain is sufficiently large. I conclude that such materials can not exist. For orders two, three, and four, stability of the rest state is possible only if the first normal stress difference has a sign incompatible with experimental values. These results extend those of Joseph [3] who showed that a fluid of grade n can be stable only if the coefficients of $\underline{A}_n$ and $\underline{A}_{n-1}$ have the same sign.

As shown in [3], the stability problem for an n^{th} grade fluid at rest leads to the characteristic equation.

$$\sigma = \Lambda_m k(\sigma) \tag{3.1}$$

$$= \Lambda_m(\alpha_0 + \alpha_1\sigma + \ldots + \alpha_{n-1}\sigma^{n-1})$$

where Λ_m are the eigenvalues of the Stokes operator for the domain occupied by the fluid and $k(\sigma)$ is a polynomial of degree $n - 1$. The Λ_m can take any positive value if the domain is chosen appropriately.

Consider the limit $\Lambda_m \to 0$ (which occurs if the domain grows large). Then one root of (3.1) tends to zero and is, in a first approximations, given by $\sigma = \alpha_0 \Lambda_m$. The other $n - 2$ roots tend to ∞ and their first approximation is $\sigma = \Lambda_m \alpha_{n-1}\sigma^{n-1}$ or $\sigma^{n-2} = \frac{1}{\Lambda_m \alpha_{n-1}}$. If n is greater than four, at least one of the roots to this last equation will have positive real part. (This argument remains valid even if an integral term as considered by Saut and Joseph [4] is added to the constitutive law. Such a term is only a lower order perturbation). If n is three or four, we conclude from the preceding argument that stability is only possible if α_0 and α_{n-1} are

negative. According to [3], α_{n-2} must then also be negative. This proves my claim for $n = 3$. For $n = 4$, let us consider the limit $\Lambda_m \to \infty$. In this case the roots are approximated by those of the equation

$$k(\sigma) = \alpha_0 + \alpha_1 \sigma + \alpha_2 \sigma^2 + \alpha_3 \sigma^3 = 0. \tag{3.2}$$

We already know that α_0, α_2, α_3 are negative. There is at least one negative root $-\sigma_0$, whence

$$\begin{aligned} k(\sigma) &= (\sigma + \sigma_0)(\beta_1 + \beta_2 \sigma + \beta_3 \sigma^2) \\ &= \beta_1 \sigma_0 + (\beta_1 + \beta_2 \sigma_0)\sigma + (\beta_2 + \beta_3 \sigma_0)\sigma^2 + \beta_3 \sigma^3. \end{aligned} \tag{3.3}$$

We can conclude that β_1 and β_3 must be negative. It is then, however, easy to see that the quadratic polynomial has a root of positive real part unless β_2 is also negative. As a consequence, $\alpha_1 = \beta_1 + \beta_2 \sigma_0$ is negative.

It must be noted that, for $n > 4$, we not only find eigenvalues of positive real part, but for $\Lambda_m \to 0$ the real part tends to infinity. In infinite domains, the initial value problem for such fluids is therefore ill-posed, i.e. solutions generally do not exist even for smooth initial data. Rivlin-Ericksen fluids of grade higher than 4 therefore are not only inadmissible as "exact" models, but also as approximate models for unsteady flows, they are an unstable approximation.

The phenomenon that an approximation to the equation does not always yield an approximation to the solution is well known in numerical analysis, however, it is by no means limited to numerical methods. Any kind of approximation must be assessed for its stability. It is my conclusion that there are two possible lines for further research:

1. Abandon the idea of retarded motion expansions [6] and seek different and hopefully stable approximation schemes.
2. Rivlin-Ericksen type approximations might be "saved" if additional approximations are introduced. This situation is quite common in numerical analysis. Many discretization methods require that spatial and temporal mesh sizes satisfy certain inequalities. This means in particular that the two approximations must not be made independently.

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